Rotation: 回転

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1 Rotation: definition and meaning

The definition of rotation of vector field $\mathbf{u}(\mathbf{r})$ is given by

$$\mathrm{rot}\mathbf{u} \equiv \nabla \times \mathbf{u} \tag{1}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u_x & u_y & u_z \end{vmatrix} \tag{2}$$

$$= \mathbf{i}(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}) + \mathbf{j}(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}) + \mathbf{k}(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}) \tag{3}$$

[1]In the vector analysis the meaning of "rot" is hard to understand, but S. Naganuma gave us a method to understand the meaning of rotation in his book (Intuitive method for phycal mathematics(Japanese))

The rotational velocity of the infinitesimal water wheel in water flow field $\mathbf{u}$. The water flow of the right side of the wheel in the upper direction u_y is faster than the left side ($\partial u_y/\partial x > 0$), the wheel rotates in the anticlockwise direction. In the same way the water flow of the upper side of the wheel in the right direction u_x is slower than the lower side ($\partial u_x/\partial y < 0$) the wheel rotates in the anticlockwise direction. Then $\partial u_y/\partial x - \partial u_x/\partial y [\equiv (\mathrm{rot}\mathbf{u})_z]$ contribute the anti. If the wheel rotates the anticlockwise direction the $(\mathrm{rot}\mathbf{u})_z$ has positive value. $(\mathrm{rot}\mathbf{u})_x$ and $(\mathrm{rot}\mathbf{u})_y$ can define in the same way.

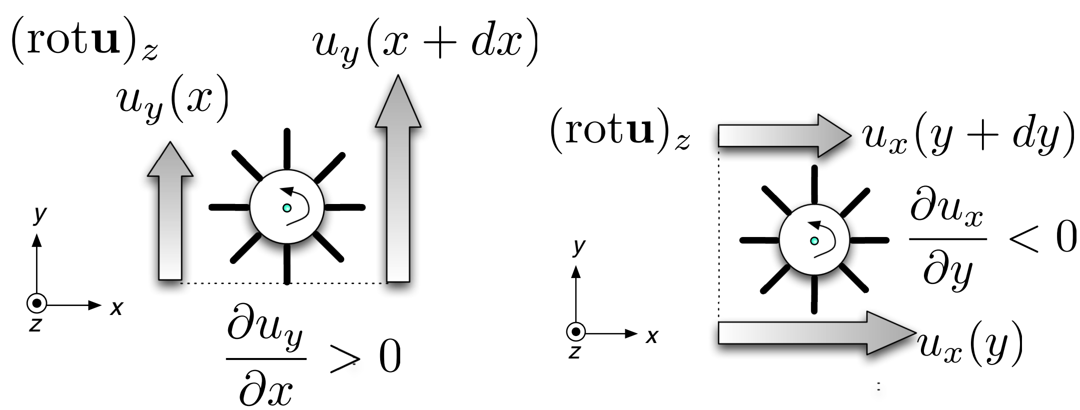

Figure 1: Naganuma's explanation of rot$\mathbf{u}$

2 Stokes theorem

As shown in Fig. 2 for a general vector field $\mathbf{V}(\mathbf{r})$, Stokes theorem gives

$$\oint \mathbf{V} \cdot d\mathbf{r} = (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}) + (\mathrm{IV})$$

[1]rot の意味はわかりにくいが、長沼伸一郎「物理数学の直感的方法」（通商産業研究社）の５章に記述されている水の流れに中においた微少な水車の回転速度という記述は大変わかりやすい。

$$\begin{aligned}
&= V_x(x_0,y_0)dx + V_y(x_0+dx,y_0)dy + V_x(x_0,y_0+dy)(-dx) + V_y(x_0,y_0)(-dy) \\
&= V_x(x_0,y_0)dx + [V_y(x_0,y_0) + \frac{\partial V_y}{\partial x}dx]dy \\
&\quad -[V_x(x_0,y_0) + \frac{\partial V_x}{\partial y}dy]dx - V_y(x_0,y_0)dy \\
&= \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right) dxdy = (\mathrm{rot}\mathbf{V})_z dxdy
\end{aligned}$$

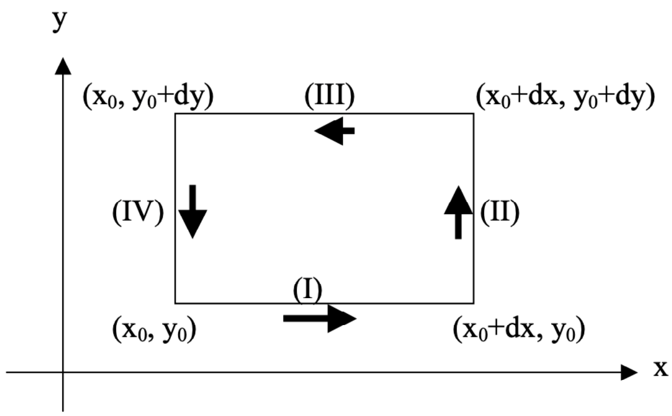

Figure 2: Stokes theorem

If we sum-up the line integral as shown in Fig.2, the integral of the neighboring contour cancels out and the contour C only survives, then we have

$$\oint_{\mathrm{C}} \mathbf{V}\cdot d\mathbf{r} \quad = \quad \int_{\mathrm{S}} \mathrm{rot}\mathbf{V}\cdot d\mathbf{S} \tag{4}$$

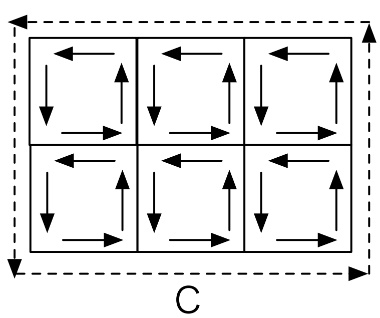

Figure 3: Integral of line and surface integral:Stokes theorem

3 Boundary Conditions at a Interface Between Different Media

Now we think a interface which is at the boundary medium 1 and 2 as shown in Fig.3. From Gauss law, we can get the following for the Gauss box which include the interface inside the box,

$$\int_V d\mathbf{r} \underbrace{\nabla\cdot\mathbf{D}}_{\rho} = \int_S \mathbf{D}\cdot d\mathbf{S} \tag{5}$$

In the limit that the Gauss box is very thin ($\delta h \to 0$)

$$\int_V d\mathbf{r}\rho = Q_{\mathrm{box}} = \mathbf{n}\cdot[\mathbf{D}_1 - \mathbf{D}_2]S \tag{6}$$

where vector $\mathbf{n}$ means the surface normal unit vector pointing from media 2 to 1.

$$\mathbf{n} \cdot [\mathbf{D}_1 - \mathbf{D}_2] = Q_{\text{box}}/S = \sigma_{12} \tag{7}$$

where σ_{12} means the interface charge density. From $\nabla \cdot \mathbf{B} = 0$

$$\mathbf{n} \cdot [\mathbf{B}_1 - \mathbf{B}_2] = 0 \tag{8}$$

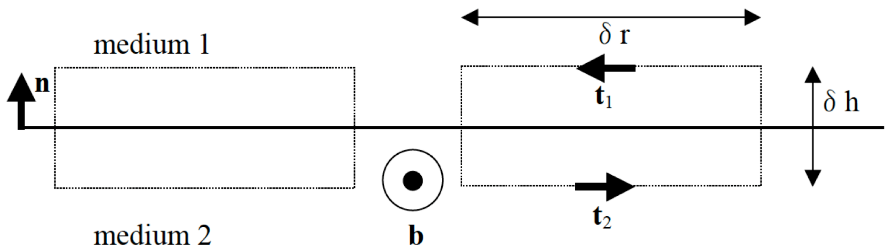

Figure 4: Gauss and Stokes box

From rot$\mathbf{E} = -\partial \mathbf{B}/\partial t$ and Stokes theorem we can get

$$\begin{aligned}
\oint_C \mathbf{E} \cdot d\mathbf{r} &= \int_S \text{rot}\mathbf{E} \cdot d\mathbf{S} && (9)\\
&= -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{b} dS && (10)\\
\delta r[\mathbf{E}_1 \cdot \mathbf{t}_1 + \mathbf{E}_2 \cdot \mathbf{t}_2] &= -\frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{b} \delta r \delta h \longrightarrow 0 && (11)\\
\mathbf{t} &= \mathbf{t}_1 = -\mathbf{t}_2 && (12)\\
\mathbf{t} \cdot [\mathbf{E}_1 - \mathbf{E}_2] &= 0 && (13)
\end{aligned}$$

Here $\mathbf{b}$ is the unit vector defined by $\mathbf{b} = \mathbf{n} \times \mathbf{t}$. From $\mathbf{rot}\mathbf{H} = \mathbf{J} + \partial \mathbf{D}/\partial t$

$$\begin{aligned}
\oint_C \mathbf{H} \cdot d\mathbf{r} &= \int_S \text{rot}\mathbf{H} \cdot d\mathbf{S} && (14)\\
&= \int_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right) \cdot \mathbf{b} dS && (15)\\
\delta r[\mathbf{H}_1 \cdot \mathbf{t}_1 + \mathbf{H}_2 \cdot \mathbf{t}_2] &= \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right) \cdot \mathbf{b} \delta r \delta h \longrightarrow \mathbf{J} \cdot \mathbf{b} \delta r \delta h && (16)\\
\mathbf{t} \cdot [\mathbf{H}_1 - \mathbf{H}_2] &= \mathbf{J}_s \quad [\equiv \text{surface current density(Am}^{-1})] && (17)
\end{aligned}$$

4 Rotation in orthogonal curvlinear coordinates

In the curvlinear coordinate

$$\begin{aligned}
\mathbf{q} &= \{q_1, q_2, q_3\} && (18)\\
\mathbf{q} &= \{x, y, z\} \quad : \text{ Cartesian coordinates} && (19)\\
\mathbf{q} &= \{r, \theta, z\} \quad : \text{ cylindircal coordinates} && (20)\\
\mathbf{q} &= \{r, \theta, \phi\} \quad : \text{ spherical coordinates} && (21)
\end{aligned}$$

If we set

$$\begin{aligned}
x &= x(q_1, q_2, q_3), \quad dx = \frac{\partial x}{\partial q_1} dq_1 + \frac{\partial x}{\partial q_2} dq_2 + \frac{\partial x}{\partial q_3} dq_3 && (22)\\
y &= y(q_1, q_2, q_3), \quad dy = \frac{\partial y}{\partial q_1} dq_1 + \frac{\partial y}{\partial q_2} dq_2 + \frac{\partial y}{\partial q_3} dq_3 && (23)\\
z &= z(q_1, q_2, q_3), \quad dz = \frac{\partial z}{\partial q_1} dq_1 + \frac{\partial z}{\partial q_2} dq_2 + \frac{\partial z}{\partial q_3} dq_3 && (24)\\
& && (25)
\end{aligned}$$

The distance ds between the two infinitesimal points is given by "metric"

$$ds^2 = dx^2 + dy^2 + dz^2 = \sum_{i,j} g_{ij} dq_i dq_j \tag{26}$$

$$g_{ij} = \frac{\partial x}{\partial q_i}\frac{\partial x}{\partial q_j} + \frac{\partial y}{\partial q_i}\frac{\partial y}{\partial q_j} + \frac{\partial z}{\partial q_i}\frac{\partial z}{\partial q_j} \tag{27}$$

$$\tag{28}$$

For orthogonal curvlinear coordinates such as cylindrical and spherical coordinates

$$g_{ij} = 0, \quad \text{for} \quad i \neq j \tag{29}$$

$$g_{ii} = \left(\frac{\partial x}{\partial q_i}\right)^2 + \left(\frac{\partial y}{\partial q_i}\right)^2 + \left(\frac{\partial z}{\partial q_i}\right)^2 \tag{30}$$

$$ds^2 = g_{11}dq_1dq_1 + g_{22}dq_2dq_2 + g_{33}dq_3dq_3 = ds_1^2 + ds_2^2 + ds_3^2 \tag{31}$$

$$ds_1 \equiv \sqrt{g_{11}}dq_1 \equiv h_1 dq_1 \tag{32}$$

$$ds_2 \equiv \sqrt{g_{22}}dq_2 \equiv h_2 dq_2 \tag{33}$$

$$ds_3 \equiv \sqrt{g_{33}}dq_3 \equiv h_3 dq_3 \tag{34}$$

example 1 : cylindrical coordinates : $x = r\cos\theta, y = r\sin\theta, z = z$

$$g_{11} = \frac{\partial x}{\partial r}\frac{\partial x}{\partial r} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial r} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial r} = \cos^2\theta + \sin^2\theta + 0 = 1, \qquad h_1 = 1 \tag{35}$$

$$g_{12} = \frac{\partial x}{\partial r}\frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial \theta} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial \theta} = \cos\theta(-r\sin\theta) + \sin\theta(r\cos\theta) + 0 = 0$$

$$g_{13} = \frac{\partial x}{\partial r}\frac{\partial x}{\partial z} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial z} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial z} = 0 + 0 + 0 = 0$$

$$g_{21} = \frac{\partial x}{\partial \theta}\frac{\partial x}{\partial r} + \frac{\partial y}{\partial \theta}\frac{\partial y}{\partial r} + \frac{\partial z}{\partial \theta}\frac{\partial z}{\partial r} = (-r\sin\theta)\cos\theta + r\cos\theta\sin\theta + 0 = 0$$

$$g_{22} = \frac{\partial x}{\partial \theta}\frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial \theta}\frac{\partial y}{\partial \theta} + \frac{\partial z}{\partial \theta}\frac{\partial z}{\partial \theta} = (-r\sin\theta)^2 + (r\cos\theta)^2 + 0 = r^2, \qquad h_2 = r \tag{36}$$

$$g_{23} = \frac{\partial x}{\partial \theta}\frac{\partial x}{\partial z} + \frac{\partial y}{\partial \theta}\frac{\partial y}{\partial z} + \frac{\partial z}{\partial \theta}\frac{\partial z}{\partial z} = 0 + 0 + 0 = 0$$

$$g_{31} = \frac{\partial x}{\partial z}\frac{\partial x}{\partial r} + \frac{\partial y}{\partial z}\frac{\partial y}{\partial r} + \frac{\partial z}{\partial z}\frac{\partial z}{\partial r} = 0 + 0 + 0 = 0$$

$$g_{32} = \frac{\partial x}{\partial z}\frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial z}\frac{\partial y}{\partial \theta} + \frac{\partial z}{\partial z}\frac{\partial z}{\partial \theta} = 0 + 0 + 0 = r^2$$

$$g_{33} = \frac{\partial x}{\partial z}\frac{\partial x}{\partial z} + \frac{\partial y}{\partial z}\frac{\partial y}{\partial z} + \frac{\partial z}{\partial z}\frac{\partial z}{\partial z} = 0 + 0 + 1^2 = 1, \qquad h_3 = 1 \tag{37}$$

example 2 : spherical coordinates : $x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi, z = r\cos\theta$

$$g_{11} = \frac{\partial x}{\partial r}\frac{\partial x}{\partial r} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial r} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial r} = \sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta = 1, \qquad h_1 = 1 \tag{38}$$

$$g_{12} = \frac{\partial x}{\partial r}\frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial \theta} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial \theta} = \sin\theta\cos\phi(r\cos\theta\cos\phi) + \sin\theta\sin\phi(r\cos\theta\sin\phi) + \cos\theta(-r\sin\theta) = 0$$

$$g_{13} = \frac{\partial x}{\partial r}\frac{\partial x}{\partial \phi} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial \phi} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial \phi} = \sin\theta\cos\phi(-r\sin\theta\sin\phi) + \sin\theta\sin\phi(r\sin\theta\cos\phi) + 0 = 0$$

$$g_{21} = \frac{\partial x}{\partial \theta}\frac{\partial x}{\partial r} + \frac{\partial y}{\partial \theta}\frac{\partial y}{\partial r} + \frac{\partial z}{\partial \theta}\frac{\partial z}{\partial r} = g_{12} = 0$$

$$g_{22} = \frac{\partial x}{\partial \theta}\frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial \theta}\frac{\partial y}{\partial \theta} + \frac{\partial z}{\partial \theta}\frac{\partial z}{\partial \theta} = (r\cos\theta\cos\phi)^2 + (r\cos\theta\sin\phi)^2 + (-r\sin\theta)^2 = r^2, \qquad h_2 = r \tag{39}$$

$$g_{23} = \frac{\partial x}{\partial \theta}\frac{\partial x}{\partial \phi} + \frac{\partial y}{\partial \theta}\frac{\partial y}{\partial \phi} + \frac{\partial z}{\partial \theta}\frac{\partial z}{\partial \phi} = r\cos\theta\cos\phi(-r\sin\theta\sin\phi) + r\cos\theta\sin\phi(r\sin\theta\cos\phi) + 0 = 0$$

$$g_{31} = \frac{\partial x}{\partial \phi}\frac{\partial x}{\partial r} + \frac{\partial y}{\partial \phi}\frac{\partial y}{\partial r} + \frac{\partial z}{\partial \phi}\frac{\partial z}{\partial r} = g_{13} = 0$$

$$g_{32} = \frac{\partial x}{\partial \phi}\frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial \phi}\frac{\partial y}{\partial \theta} + \frac{\partial z}{\partial \phi}\frac{\partial z}{\partial \theta} = g_{23} = 0$$

$$g_{33} = \frac{\partial x}{\partial \phi}\frac{\partial x}{\partial \phi} + \frac{\partial y}{\partial \phi}\frac{\partial y}{\partial \phi} + \frac{\partial z}{\partial \phi}\frac{\partial z}{\partial \phi} = (-r\sin\theta\sin\phi)^2 + (r\sin\theta\cos\phi)^2 + 0 = r^2\sin^2\theta, \qquad h_3 = r\sin\theta \quad (40)$$

The differential distance vector $d\mathbf{r}$ and the unit vector $\mathbf{e}_i$ may be given

$$d\mathbf{r} = h_1 dq_1 \mathbf{e}_1 + h_2 dq_2 \mathbf{e}_2 + h_3 dq_3 \mathbf{e}_3 = \sum_{i=1}^{3} h_i q_i \mathbf{e}_i \quad (41)$$

$$\mathbf{e}_i = \frac{1}{h_i}\frac{\partial \mathbf{r}}{\partial q_i} \quad (42)$$

Here we only consider orthogonal curvlinear coordinates such as cylindrical coordinate and spherical coordinates.

4.1 gradient

The gradient can be defined by the sum of the product of the slope (rate of change) and the unit vector in the i-direction. The slope can be written as

$$\frac{\partial f}{\partial s_i} = \frac{1}{h_i}\frac{\partial f}{\partial q_i} \quad (43)$$

$$(44)$$

Then the gradient in the orthogonal curvlinear coordinates

$$\mathrm{grad}_{\mathbf{q}} f = \nabla_{\mathbf{q}} f = \mathbf{e}_1 \frac{\partial f}{\partial s_1} + \mathbf{e}_2 \frac{\partial f}{\partial s_2} + \mathbf{e}_3 \frac{\partial f}{\partial s_1} \quad (45)$$

$$= \mathbf{e}_1 \frac{1}{h_1}\frac{\partial f}{\partial q_1} + \mathbf{e}_2 \frac{1}{h_2}\frac{\partial f}{\partial q_2} + \mathbf{e}_3 \frac{1}{h_3}\frac{\partial f}{\partial q_3} \quad (46)$$

$$\mathbf{e}_1 = \frac{1}{h_1}\frac{\partial \mathbf{r}}{\partial q_1}, \quad \mathbf{e}_2 = \frac{1}{h_2}\frac{\partial \mathbf{r}}{\partial q_2}, \quad \mathbf{e}_3 = \frac{1}{h_3}\frac{\partial \mathbf{r}}{\partial q_3} \quad (47)$$

In the cartesian coordinates,

$$\mathbf{q} = \{x, y, z\}, \quad \mathbf{r} = (x, y, z) \quad (48)$$

$$h_1 = 1, \quad \frac{\partial \mathbf{r}}{\partial x} = (1, 0, 0), \quad \mathbf{e}_1 = (1, 0, 0) = \mathbf{i} \quad (49)$$

$$h_2 = 1, \quad \frac{\partial \mathbf{r}}{\partial y} = (0, 1, 0), \quad \mathbf{e}_1 = (0, 1, 0) = \mathbf{j} \quad (50)$$

$$h_3 = 1, \quad \frac{\partial \mathbf{r}}{\partial z} = (0, 0, 1), \quad \mathbf{e}_1 = (0, 0, 1) = \mathbf{k} \quad (51)$$

$$\mathrm{grad} f = \nabla f = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) f \quad (52)$$

In the cylindrical coordinates,

$$\mathbf{q} = \{r, \theta, z\}, \quad \mathbf{r} = (x, y, z) = (r\cos\theta, r\sin\theta, z) \quad (53)$$

$$\frac{\partial \mathbf{r}}{\partial r} = (\cos\theta, \sin\theta, 0), \quad h_1 = \left|\frac{\partial \mathbf{r}}{\partial r}\right| = \sqrt{\cos^2\theta + \sin^2\theta} = 1, \quad \mathbf{e}_1 = (\cos\theta, \sin\theta, 0) \quad (54)$$

$$\frac{\partial \mathbf{r}}{\partial \theta} = (-r\sin\theta, r\cos\theta, 0), \quad h_2 = \left|\frac{\partial \mathbf{r}}{\partial \theta}\right| = r\sqrt{\cos^2\theta + \sin^2\theta} = r, \quad \mathbf{e}_2 = (-\sin\theta, \cos\theta, 0) \quad (55)$$

$$\frac{\partial \mathbf{r}}{\partial z} = (0, 0, 1), \quad h_3 = \left|\frac{\partial \mathbf{r}}{\partial z}\right| = 1, \quad \mathbf{e}_3 = (0, 0, 1) \quad (56)$$

$$\mathrm{grad}_{r,\theta,z} f = \nabla_{r,\theta,z} = \left(\mathbf{e}_1\frac{\partial}{\partial r} + \mathbf{e}_2\frac{1}{r}\frac{\partial}{\partial \theta} + \mathbf{e}_3\frac{\partial}{\partial z}\right) f \quad (57)$$

In the spherical coordinates,

$$\mathbf{q} = \{r, \theta, \phi\}, \quad \mathbf{r} = (x, y, z) = (r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta) \quad (58)$$

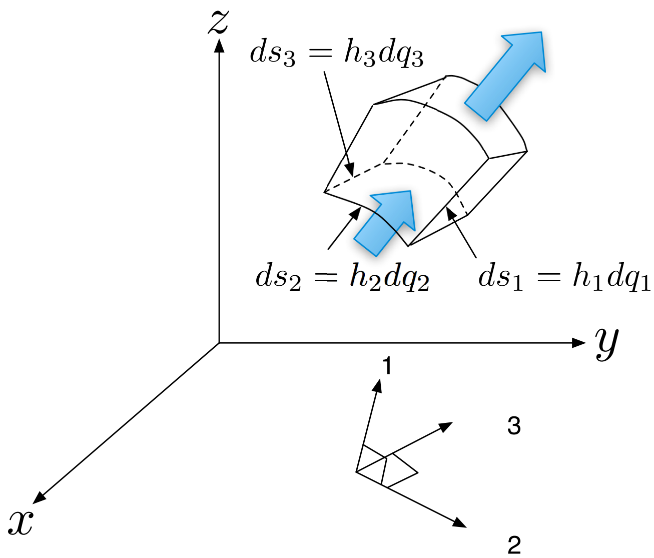

Figure 5:

$$\begin{aligned}
\frac{\partial \mathbf{r}}{\partial r} &= (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \quad h_1 = \left|\frac{\partial \mathbf{r}}{\partial r}\right| = \sqrt{\sin^2\theta(\cos^2\phi + \sin^2\phi) + \cos^2\theta} = 1, \\
&\mathbf{e}_1 = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \quad (59) \\
\frac{\partial \mathbf{r}}{\partial \theta} &= (r\cos\theta\cos\phi, r\cos\theta\sin\phi, -r\sin\theta), \quad h_2 = \left|\frac{\partial \mathbf{r}}{\partial \theta}\right| = r\sqrt{\cos^2\theta(\cos^2\phi + \sin^2\phi) + \sin^2\theta} = r \\
&\mathbf{e}_2 = (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta) \quad (60) \\
\frac{\partial \mathbf{r}}{\partial \phi} &= (-r\sin\theta\sin\phi, r\sin\theta\cos\phi, 0), \quad h_3 = \left|\frac{\partial \mathbf{r}}{\partial \phi}\right| = r\sqrt{\sin^2\theta\sin^2\phi + \sin^2\theta\cos^2\phi} = r\sin\theta, \\
&\mathbf{e}_3 = (-\sin\phi, \cos\phi, 0) \quad (61) \\
\mathrm{grad}_{r,\theta,\phi} f &= \nabla_{r,\theta,\phi} = \left(\mathbf{e}_1\frac{\partial}{\partial r} + \mathbf{e}_2\frac{1}{r}\frac{\partial}{\partial\theta} + \mathbf{e}_3\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right) f \quad (62)
\end{aligned}$$

4.2 divergence

We will consider the flux vector $\mathbf{J}$ in Fig.3. The mass balance of the flow-in and flow-out in the J_1 direction can be obtained if we consider the flow-out area also depends on q_1 [2]

$$\left[J_1 ds_2 ds_3 + \frac{\partial(J_1 ds_2 ds_3)}{\partial q_1} dq_1\right] - J_1 ds_2 ds_3 = \frac{\partial(J_1 h_2 h_3)}{\partial q_1} dq_1 dq_2 dq_3 \quad (63)$$

In the same way we have the total amount for the volume element $ds_1 ds_2 ds_3$

$$\left[\frac{\partial(J_1 h_2 h_3)}{\partial q_1} + \frac{\partial(J_2 h_3 h_1)}{\partial q_2} + \frac{\partial(J_3 h_1 h_2)}{\partial q_3}\right] dq_1 dq_2 dq_3 \quad (64)$$

In the same way as the divergence in x, y, z-coordinate, the divergence is given by the divide of the unit volume

$$\begin{aligned}
\mathrm{div}\mathbf{J}(q_1, q_2, q_3) &= \nabla\cdot\mathbf{J}(q_1, q_2, q_3) = \frac{1}{ds_1 ds_2 ds_3}\left[\frac{\partial(J_1 h_2 h_3)}{\partial q_1} + \frac{\partial(J_2 h_3 h_1)}{\partial q_2} + \frac{\partial(J_3 h_1 h_2)}{\partial q_3}\right] dq_1 dq_2 dq_3 \quad (65) \\
&= \frac{1}{h_1 h_2 h_3}\left[\frac{\partial(J_1 h_2 h_3)}{\partial q_1} + \frac{\partial(J_2 h_3 h_1)}{\partial q_2} + \frac{\partial(J_3 h_1 h_2)}{\partial q_3}\right] \quad (66) \\
J_1 &= \mathbf{J}\cdot\mathbf{e}_1, \quad J_2 = \mathbf{J}\cdot\mathbf{e}_2, \quad J_3 = \mathbf{J}\cdot\mathbf{e}_3 \quad (67)
\end{aligned}$$

[2] The expansion may be done by s_i but the differentiation should be done by q_i. So the approximation is used here.

In the cartesian coordinate

$$\begin{aligned}
\mathbf{q} &= (x, y, z) && (68)\\
h_1 &= 1, \quad h_2 = 1, \quad h_3 = 1 && (69)\\
\mathrm{div}\mathbf{J}(x, y, z) &= \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} && (70)
\end{aligned}$$

In the cylindrical coordinates

$$\begin{aligned}
\mathbf{q} &= (r, \theta, z) && (71)\\
h_1 &= 1, \quad h_2 = r, \quad h_3 = 1 && (72)\\
\mathrm{div}_{r,\theta,z}\mathbf{J} = \nabla_{r,\theta,z} \cdot \mathbf{J} &= \frac{1}{r}\left[\frac{\partial(rJ_r)}{\partial r} + \frac{\partial J_\theta}{\partial \theta} + \frac{\partial(rJ_z)}{\partial z}\right] && (73)\\
&= \frac{1}{r}\left[\frac{\partial(rJ_r)}{\partial r} + \frac{\partial J_\theta}{\partial \theta} + r\frac{\partial J_z}{\partial z}\right] && (74)
\end{aligned}$$

In the spherical coordinates

$$\begin{aligned}
\mathbf{q} &= (r, \theta, \phi) && (75)\\
h_1 &= 1, \quad h_2 = r, \quad h_3 = r\sin\theta && (76)\\
\mathrm{div}_{r,\theta,\phi}\mathbf{J} = \nabla_{r,\theta,\phi} \cdot \mathbf{J} &= \frac{1}{r^2\sin\theta}\left[\frac{\partial(r^2\sin\theta J_r)}{\partial r} + \frac{\partial(r\sin\theta J_\theta)}{\partial \theta} + \frac{\partial(rJ_\phi)}{\partial \phi}\right] && (77)\\
&= \frac{1}{r^2\sin\theta}\left[\sin\theta\frac{\partial(r^2 J_r)}{\partial r} + r\frac{\partial(\sin\theta J_\theta)}{\partial \theta} + r\frac{\partial J_\phi}{\partial \phi}\right] && (78)\\
& && (79)
\end{aligned}$$

4.3 Laplacian

We also get Laplacian ∇^2 when we use

$$\begin{aligned}
\mathbf{J}(q_1, q_2, q_3) &= \mathrm{grad}_{\mathbf{q}} f = \mathbf{e}_1\frac{1}{h_1}\frac{\partial f}{\partial q_1} + \mathbf{e}_2\frac{1}{h_2}\frac{\partial f}{\partial q_2} + \mathbf{e}_3\frac{1}{h_3}\frac{\partial f}{\partial q_3} && (80)\\
\mathrm{div}\mathbf{J}(q_1, q_2, q_3) &= \nabla \cdot \mathbf{J}(q_1, q_2, q_3) = \mathrm{div}(\mathrm{grad}_{\mathbf{q}} f) = \nabla^2_{\mathbf{q}} f && (81)\\
\nabla^2_{\mathbf{q}} f &= \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2h_3}{h_1}\frac{\partial f}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_3h_1}{h_2}\frac{\partial f}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{h_1h_2}{h_3}\frac{\partial f}{\partial q_3}\right)\right] && (82)
\end{aligned}$$

The Laplacian is very important in physical chemistry.
(a) For the diffusion equation

$$\begin{aligned}
\frac{\partial c}{\partial t} + \mathrm{div}\mathbf{J} &= 0, \quad \mathbf{J} = -\mathrm{grad}c\\
\frac{\partial c}{\partial t} &= \mathrm{div}(\mathrm{grad}c) = \nabla^2 c
\end{aligned}$$

(b) For the wave equation

$$\frac{\partial^2 \xi}{\partial t^2} = v^2\mathrm{div}(\mathrm{grad}\xi) = v^2\nabla^2\xi$$

In the cartesian cordinate

$$\begin{aligned}
\mathbf{q} &= \{x, y, z\}\\
h_1 &= 1, \quad h_2 = 1, \quad h_3 = 1\\
\nabla^2_{\mathbf{q}} f &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}
\end{aligned}$$

In the cylindrical coordinates

$$\begin{aligned}
\mathbf{q} &= \{r, \theta, z\} \\
h_1 &= 1, \quad h_2 = r, \quad h_3 = 1 \\
\nabla^2_{\mathbf{q}} f &= \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial f}{\partial \theta}\right) + \frac{\partial}{\partial z}\left(r\frac{\partial f}{\partial z}\right)\right] \\
&= \frac{1}{r}\frac{\partial f}{\partial r} + \frac{\partial^2 f}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}
\end{aligned}$$

In the spherical coordinates

$$\mathbf{q} = (r, \theta, \phi) \tag{83}$$

$$h_1 = 1, \quad h_2 = r, \quad h_3 = r\sin\theta \tag{84}$$

$$\nabla^2_{\mathbf{q}} f = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial f}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\frac{r\sin\theta}{r}\frac{\partial f}{\partial \theta}\right) + \frac{\partial}{\partial \phi}\left(\frac{r}{r\sin\theta}\frac{\partial f}{\partial \phi}\right)\right] \tag{85}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2} \tag{86}$$

$$= \frac{2}{r}\frac{\partial f}{\partial r} + \frac{\partial^2 f}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\cos\theta}{r^2\sin\theta}\frac{\partial f}{\partial \theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2} \tag{87}$$

4.4 rotation

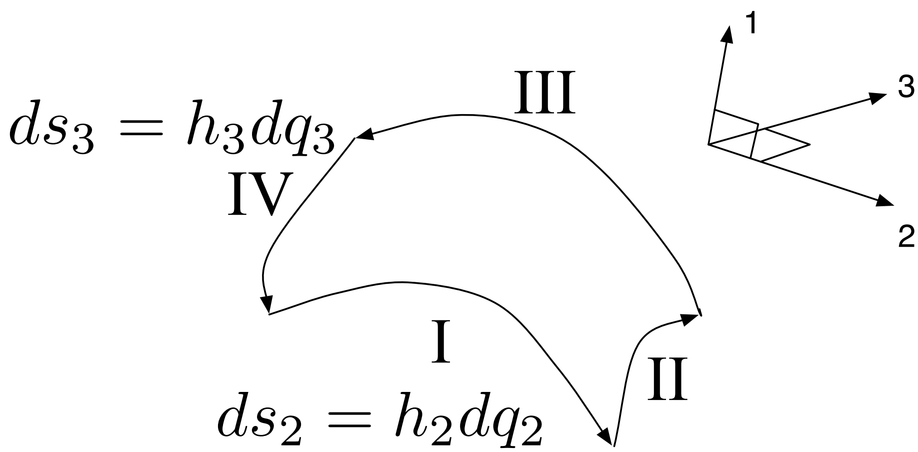

Figure 6: differential surface element in the curvlinear surface q_1 = constant.

We will apply the Stokes theorem

$$\int_S \mathrm{rot}\mathbf{J}\cdot d\mathbf{S} = \oint_C \mathbf{J}\cdot\mathbf{r} \tag{88}$$

For the component q_1 direction (as shown in Fig.4.4) the Stokes theorem tells us

$$\int_S (\mathrm{rot}\mathbf{J})_1 (d\mathbf{S})_1 = (\mathrm{rot}\mathbf{J})_1 ds_2 ds_3 = (\mathrm{rot}\mathbf{J})_1 h_2 h_3 dq_2 dq_3 \tag{89}$$

$$\oint_C \mathbf{J}\cdot\mathbf{r} = \underbrace{J_2 h_2 dq_2}_{\text{path I}} + \underbrace{\left(J_3 h_3 dq_3 + \frac{\partial(J_3 h_3)}{\partial q_2} dq_2 dq_3\right)}_{\text{path II}} - \underbrace{\left(J_2 h_2 dq_2 + \frac{\partial(J_2 h_2)}{\partial q_3} dq_3 dq_2\right)}_{\text{path III}} \underbrace{-J_3 h_3 dq_3}_{\text{path IV}}$$

$$= \frac{\partial(J_3 h_3)}{\partial q_2} dq_2 dq_3 - \frac{\partial(J_2 h_2)}{\partial q_3} dq_2 dq_3 \tag{90}$$

$$(\mathrm{rot}\mathbf{J})_1 = \frac{1}{h_2 h_3}\left[\frac{\partial(J_3 h_3)}{\partial q_2} - \frac{\partial(J_2 h_2)}{\partial q_3}\right] \tag{91}$$

$$\begin{aligned}
\mathrm{rot}\mathbf{J} &\equiv \nabla_{\mathbf{q}} \times \mathbf{J} &(92)\\
&= \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1\mathbf{e}_1 & h_2\mathbf{e}_2 & h_3\mathbf{e}_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ J_1 h_1 & J_2 h_2 & J_3 h_3 \end{vmatrix} &(93)\\
&= \mathbf{e}_1 \frac{1}{h_2 h_3}\left[\frac{\partial(J_3 h_3)}{\partial q_2} - \frac{\partial(J_2 h_2)}{\partial q_3}\right] \\
&\quad + \mathbf{e}_2 \frac{1}{h_3 h_1}\left[\frac{\partial(J_1 h_1)}{\partial q_3} - \frac{\partial(J_3 h_3)}{\partial q_1}\right] \\
&\quad + \mathbf{e}_3 \frac{1}{h_1 h_2}\left[\frac{\partial(J_2 h_2)}{\partial q_1} - \frac{\partial(J_1 h_1)}{\partial q_2}\right] &(94)
\end{aligned}$$

In the cartesian coordinates,

$$\begin{aligned}
\mathbf{q} &= \{x, y, z\}, \quad \mathbf{r} = (x, y, z) &(95)\\
h_1 &= 1, \mathbf{e}_1 = (1, 0, 0) = \mathbf{i} &(96)\\
h_2 &= 1, \mathbf{e}_2 = (0, 1, 0) = \mathbf{j} &(97)\\
h_3 &= 1, \mathbf{e}_3 = (0, 0, 1) = \mathbf{k} &(98)\\
\mathrm{rot}\mathbf{J} &= \nabla \times \mathbf{J} = \mathbf{i}\left(\frac{\partial J_z}{\partial y} - \frac{\partial J_y}{\partial z}\right) + \mathbf{j}\left(\frac{\partial J_x}{\partial z} - \frac{\partial J_z}{\partial x}\right) + \mathbf{k}\left(\frac{\partial J_y}{\partial x} - \frac{\partial J_x}{\partial y}\right) &(99)
\end{aligned}$$

In the cylindrical coordinates,

$$\begin{aligned}
\mathbf{q} &= \{r, \theta, z\}, \quad \mathbf{r} = (x, y, z) = (r\cos\theta, r\sin\theta, z) &(100)\\
h_1 &= 1, \quad \mathbf{e}_1 = (\cos\theta, \sin\theta, 0) &(101)\\
h_2 &= r, \quad \mathbf{e}_2 = (-\sin\theta, \cos\theta, 0) &(102)\\
h_3 &= 1, \quad \mathbf{e}_3 = (0, 0, 1) &(103)\\
\mathrm{rot}\mathbf{J} &= \nabla \times \mathbf{J} = \mathbf{e}_1 \frac{1}{r}\left(\frac{\partial J_z}{\partial \theta} - \frac{\partial (rJ_\theta)}{\partial z}\right) + \mathbf{e}_2\left(\frac{\partial J_r}{\partial z} - \frac{\partial J_z}{\partial r}\right) + \mathbf{e}_3 \frac{1}{r}\left(\frac{\partial (rJ_\theta)}{\partial r} - \frac{\partial J_r}{\partial \theta}\right) &(104)
\end{aligned}$$

In the spherical coordinates

$$\begin{aligned}
\mathbf{q} &= \{r, \theta, \phi\}, \quad \mathbf{r} = (x, y, z) = (r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta) &(105)\\
h_1 &= 1, \quad \mathbf{e}_1 = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) &(106)\\
h_2 &= r, \quad \mathbf{e}_2 = (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta) &(107)\\
h_3 &= r\sin\theta, \quad \mathbf{e}_3 = (-\sin\phi, \cos\phi, 0) &(108)\\
\mathrm{rot}\mathbf{J} &= \nabla \times \mathbf{J} \\
&= \mathbf{e}_1 \frac{1}{r^2\sin\theta}\left(\frac{\partial (r\sin\theta J_\phi)}{\partial \theta} - \frac{\partial (rJ_\theta)}{\partial \phi}\right) + \mathbf{e}_2 \frac{1}{r\sin\theta}\left(\frac{\partial J_r}{\partial \phi} - \frac{\partial (r\sin\theta J_\phi)}{\partial r}\right) + \mathbf{e}_3 \frac{1}{r^2\sin\theta}\left(\frac{\partial (rJ_\theta)}{\partial r} - \frac{\partial J_r}{\partial \theta}\right) &(109)
\end{aligned}$$