# Gradient: 勾配（こうばい）

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## 1 definition of grad

The gradient of the scalar function $\phi(\mathbf{r})$ is defined by

$$\mathrm{grad}\phi = \nabla\phi(\mathbf{r}) = \mathbf{i}\frac{\partial\phi}{\partial x} + \mathbf{j}\frac{\partial\phi}{\partial y} + \mathbf{k}\frac{\partial\phi}{\partial z} \tag{1}$$

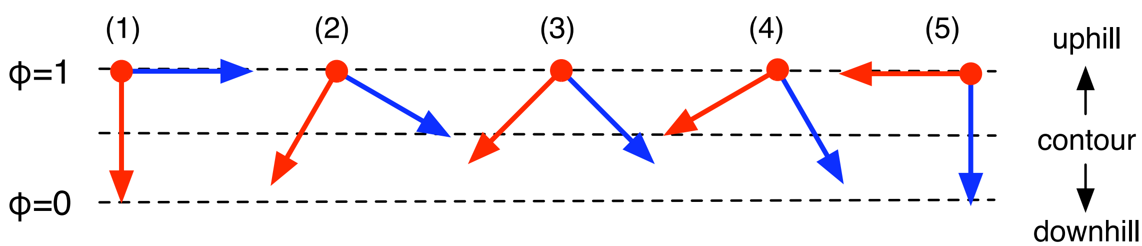


Figure 1: Gradient of the scalar function $\phi$ in 2D case. The red vector is the unit vector $\mathbf{i}$(x-axis) and the blue vector is the unit vector $\mathbf{j}$(y-axis). The $x - y$ axis is rotated by (1) 0, (2) 30, (3) 45, (4) 60, and (5) 90 degree in the clockwise direction, respectively.

In (1), $\phi = 1 - x$, $\nabla\phi = -\mathbf{i}$. In (2), $\phi = 1 - \frac{\sqrt{3}}{2}x - \frac{1}{2}y$, $\nabla\phi = -\frac{\sqrt{3}}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}$. In (3), $\phi = 1 - \frac{1}{\sqrt{2}}x - \frac{1}{\sqrt{2}}y$, $\nabla\phi = -\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}$. In (4), $\phi = 1 - \frac{1}{2}x - \frac{\sqrt{3}}{2}y$, $\nabla\phi = -\frac{1}{2}\mathbf{i} - \frac{\sqrt{3}}{2}\mathbf{j}$. In (5), $\phi = 1 - y$, $\nabla\phi = -\mathbf{j}$. These all show the minus of the direction of the gradient is in the steepest descent one.

Now we define that the line from $\mathbf{r}$ to $\mathbf{r} + \mathrm{d}\mathbf{c}$ is along the contour line(2D) or hypersurface (3D), then

$$d\phi = \phi(\mathbf{r} + d\mathbf{c}) - \phi(\mathbf{r}) = 0 \tag{2}$$

$$\begin{aligned} d\phi &= \phi(\mathbf{r}) + \frac{d\phi}{d\mathbf{r}}d\mathbf{c} - \phi(\mathbf{r}) \\ &= \nabla\phi \cdot d\mathbf{c} = 0 \end{aligned} \tag{3}$$

Then we showed that the gradient direction is normal to the contour line, i.e. the steepest descent direction.

Next we define that the line from $\mathbf{r}$ to $\mathbf{r} + \mathrm{d}\mathbf{s}$ is along the steepest descent direction,

$$d\phi = \phi(\mathbf{r} + d\mathbf{s}) - \phi(\mathbf{r}) \tag{4}$$

$$\begin{aligned} &= \phi(\mathbf{r}) + \frac{d\phi}{d\mathbf{r}}d\mathbf{s} - \phi(\mathbf{r}) \\ &= \nabla\phi \cdot d\mathbf{s} = |\nabla\phi| ds \end{aligned} \tag{5}$$

$$\frac{d\phi}{ds} = |\nabla\phi| \tag{6}$$

Then we showed the slope in the steepest descent direction is given by grad$\phi$. The norm of the gradient is given by

$$|\mathrm{grad}\phi| = \sqrt{\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2} \tag{7}$$

## 2 vector field and potential

Now we consider the electric field. The point-charge with charge $q$ is moved in the electric filed $\mathbf{E}(\mathbf{r})$. The force $\mathbf{F}(\mathbf{r})$ which act on the point-charge from the field is given by $\mathbf{F}(\mathbf{r}) = q\mathbf{E}(\mathbf{r})$, The energy difference from point O to P is

$$U(\mathrm{O} \to \mathrm{P}) = -\int_{\mathrm{O}}^{\mathrm{P}} \mathbf{F} \cdot d\mathbf{s} = -q \int_{\mathrm{O}}^{\mathrm{P}} \mathbf{E} \cdot d\mathbf{s} \tag{8}$$

Here $d\mathbf{s}$ is taken along the contour form O to P. The potential $\phi$ is defined by

$$\phi(P) - \phi(O) = \frac{U(\mathrm{O} \to \mathrm{P})}{q} = = -\int_{\mathrm{O}}^{\mathrm{P}} \mathbf{E} \cdot d\mathbf{s} \tag{9}$$

If P is at $\mathbf{r} + d\mathbf{s}$ and O is at $\mathbf{r}$

$$\phi(\mathbf{r} + d\mathbf{s}) - \phi(\mathbf{r}) = \frac{d\phi}{d\mathbf{r}} d\mathbf{s} = \nabla\phi \cdot d\mathbf{s} = -\mathbf{E} \cdot d\mathbf{s} \tag{10}$$

Then we have

$$\mathbf{E} = -\nabla\phi \tag{11}$$

This means the electric field is along the steepest-descent direction of potential.

If we remind that $\nabla \cdot \mathbf{E} = \rho/(\epsilon_0\epsilon)$, we have Poisson equation

$$\mathrm{div}(-\mathrm{grad}\phi) = \frac{\rho}{\epsilon_0\epsilon} \tag{12}$$

$$-\nabla^2\phi = \frac{\rho}{\epsilon_0\epsilon} \tag{13}$$

## 3 1D problem of Poisson equation

When we know the charge density $\rho(z)$ and $\rho(-\infty) = d\rho(-\infty)/dz = 0$, we can calculate the potential $\phi(z)$

$$\phi(z) = -\frac{1}{\epsilon_0\epsilon} \int_{-\infty}^{z} (z - z')\rho(z')dz' \tag{14}$$

This equation satisfy the Poisson equation, because

$$\phi(z) = -\frac{1}{\epsilon_0\epsilon} z \int_{-\infty}^{z} \rho(z')dz' + \frac{1}{\epsilon_0\epsilon} \int_{-\infty}^{z} z'\rho(z')dz' \tag{15}$$

$$\frac{d\phi}{dz} = -\frac{1}{\epsilon_0\epsilon} \int_{-\infty}^{z} \rho(z')dz' - \frac{1}{\epsilon_0\epsilon} z\rho(z) + \frac{1}{\epsilon_0\epsilon} z\rho(z) = -\frac{1}{\epsilon_0\epsilon} \int_{-\infty}^{z} \rho(z')dz' \tag{16}$$

$$\frac{d^2\phi}{dz^2} = -\frac{1}{\epsilon_0\epsilon}\rho(z) \tag{17}$$

Here we used

$$\frac{dR}{dz} = \rho(z), \quad \frac{d}{dz} \int_{-\infty}^{z} \rho(z')dz' = \frac{d}{dz}[R(z) - R(-\infty)] = \rho(z) - \rho(-\infty) = \rho(z) \tag{18}$$

$$\frac{dQ}{dz} = z\rho(z), \quad \frac{d}{dz} \int_{-\infty}^{z} z'\rho(z')dz' = \frac{d}{dz}[Q(z) - Q(-\infty)] = z\rho(z) - (-\infty)\rho(-\infty) = z\rho(z) \tag{19}$$

In the same way when we know the charge density $\rho(z)$ and $\rho(+\infty) = d\rho(+\infty)/dz = 0$,

$$\phi(z) = \frac{1}{\epsilon_0\epsilon} \int_{z}^{+\infty} (z - z')\rho(z')dz' \tag{20}$$

## 4 gradient in curvlinear coordinate: 曲線座標系の勾配

In the curvlinear coordinate

$$\mathbf{q} = \{q_1, q_2, q_3\} \tag{21}$$
$$\mathbf{q} = \{x, y, z\} \quad : \text{ Cartesian coordinates} \tag{22}$$
$$\mathbf{q} = \{r, \theta, z\} \quad : \text{ cylindircal coordinates} \tag{23}$$
$$\mathbf{q} = \{r, \theta, \phi\} \quad : \text{ spherical coordinates} \tag{24}$$

If we set

$$x = x(q_1, q_2, q_3), \quad dx = \frac{\partial x}{\partial q_1}dq_1 + \frac{\partial x}{\partial q_2}dq_2 + \frac{\partial x}{\partial q_3}dq_3 \tag{25}$$
$$y = y(q_1, q_2, q_3), \quad dy = \frac{\partial y}{\partial q_1}dq_1 + \frac{\partial y}{\partial q_2}dq_2 + \frac{\partial y}{\partial q_3}dq_3 \tag{26}$$
$$z = z(q_1, q_2, q_3), \quad dz = \frac{\partial z}{\partial q_1}dq_1 + \frac{\partial z}{\partial q_2}dq_2 + \frac{\partial z}{\partial q_3}dq_3 \tag{27}$$
$$\tag{28}$$

The distance $ds$ between the two infinitesimal points is given by "metric"

$$ds^2 = dx^2 + dy^2 + dz^2 = \sum_{i,j} g_{ij} dq_i dq_j \tag{29}$$
$$g_{ij} = \frac{\partial x}{\partial q_i}\frac{\partial x}{\partial q_j} + \frac{\partial y}{\partial q_i}\frac{\partial y}{\partial q_j} + \frac{\partial z}{\partial q_i}\frac{\partial z}{\partial q_j} \tag{30}$$
$$\tag{31}$$

For orthogonal curvlinear coordinates such as cylindrical and spherical coordinates

$$g_{ij} = 0, \quad \text{for} \quad i \neq j \tag{32}$$
$$g_{ii} = \left(\frac{\partial x}{\partial q_i}\right)^2 + \left(\frac{\partial y}{\partial q_i}\right)^2 + \left(\frac{\partial z}{\partial q_i}\right)^2 \tag{33}$$
$$ds^2 = g_{11}dq_1dq_1 + g_{22}dq_2dq_2 + g_{33}dq_3dq_3 = ds_1^2 + ds_2^2 + ds_3^2 \tag{34}$$
$$ds_1 \equiv \sqrt{g_{11}}dq_1 \equiv h_1 dq_1 \tag{35}$$
$$ds_2 \equiv \sqrt{g_{22}}dq_2 \equiv h_2 dq_2 \tag{36}$$
$$ds_3 \equiv \sqrt{g_{33}}dq_3 \equiv h_3 dq_3 \tag{37}$$

example 1 : cylindrical coordinates : $x = r\cos\theta, y = r\sin\theta, z = z$

$$g_{11} = \frac{\partial x}{\partial r}\frac{\partial x}{\partial r} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial r} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial r} = \cos^2\theta + \sin^2\theta + 0 = 1, \qquad h_1 = 1 \tag{38}$$
$$g_{12} = \frac{\partial x}{\partial r}\frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial \theta} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial \theta} = \cos\theta(-r\sin\theta) + \sin\theta(r\cos\theta) + 0 = 0$$
$$g_{13} = \frac{\partial x}{\partial r}\frac{\partial x}{\partial z} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial z} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial z} = 0 + 0 + 0 = 0$$
$$g_{21} = \frac{\partial x}{\partial \theta}\frac{\partial x}{\partial r} + \frac{\partial y}{\partial \theta}\frac{\partial y}{\partial r} + \frac{\partial z}{\partial \theta}\frac{\partial z}{\partial r} = (-r\sin\theta)\cos\theta + r\cos\theta\sin\theta + 0 = 0$$
$$g_{22} = \frac{\partial x}{\partial \theta}\frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial \theta}\frac{\partial y}{\partial \theta} + \frac{\partial z}{\partial \theta}\frac{\partial z}{\partial \theta} = (-r\sin\theta)^2 + (r\cos\theta)^2 + 0 = r^2, \qquad h_2 = r \tag{39}$$
$$g_{23} = \frac{\partial x}{\partial \theta}\frac{\partial x}{\partial z} + \frac{\partial y}{\partial \theta}\frac{\partial y}{\partial z} + \frac{\partial z}{\partial \theta}\frac{\partial z}{\partial z} = 0 + 0 + 0 = 0$$
$$g_{31} = \frac{\partial x}{\partial z}\frac{\partial x}{\partial r} + \frac{\partial y}{\partial z}\frac{\partial y}{\partial r} + \frac{\partial z}{\partial z}\frac{\partial z}{\partial r} = 0 + 0 + 0 = 0$$
$$g_{32} = \frac{\partial x}{\partial z}\frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial z}\frac{\partial y}{\partial \theta} + \frac{\partial z}{\partial z}\frac{\partial z}{\partial \theta} = 0 + 0 + 0 = r^2$$

$$g_{33} = \frac{\partial x}{\partial z}\frac{\partial x}{\partial z} + \frac{\partial y}{\partial z}\frac{\partial y}{\partial z} + \frac{\partial z}{\partial z}\frac{\partial z}{\partial z} = 0 + 0 + 1^2 = 1, \qquad h_3 = 1 \tag{40}$$

example 2 : spherical coordinates : $x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi, z = r\cos\theta$

$$g_{11} = \frac{\partial x}{\partial r}\frac{\partial x}{\partial r} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial r} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial r} = \sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta = 1, \qquad h_1 = 1 \tag{41}$$

$$g_{12} = \frac{\partial x}{\partial r}\frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial \theta} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial \theta} = \sin\theta\cos\phi(r\cos\theta\cos\phi) + \sin\theta\sin\phi(r\cos\theta\sin\phi) + \cos\theta(-r\sin\theta) = 0$$

$$g_{13} = \frac{\partial x}{\partial r}\frac{\partial x}{\partial \phi} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial \phi} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial \phi} = \sin\theta\cos\phi(-r\sin\theta\sin\phi) + \sin\theta\sin\phi(r\sin\theta\cos\phi) + 0 = 0$$

$$g_{21} = \frac{\partial x}{\partial \theta}\frac{\partial x}{\partial r} + \frac{\partial y}{\partial \theta}\frac{\partial y}{\partial r} + \frac{\partial z}{\partial \theta}\frac{\partial z}{\partial r} = g_{12} = 0$$

$$g_{22} = \frac{\partial x}{\partial \theta}\frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial \theta}\frac{\partial y}{\partial \theta} + \frac{\partial z}{\partial \theta}\frac{\partial z}{\partial \theta} = (r\cos\theta\cos\phi)^2 + (r\cos\theta\sin\phi)^2 + (-r\sin\theta)^2 = r^2, \qquad h_2 = r \tag{42}$$

$$g_{23} = \frac{\partial x}{\partial \theta}\frac{\partial x}{\partial \phi} + \frac{\partial y}{\partial \theta}\frac{\partial y}{\partial \phi} + \frac{\partial z}{\partial \theta}\frac{\partial z}{\partial \phi} = r\cos\theta\cos\phi(-r\sin\theta\sin\phi) + r\cos\theta\sin\phi(r\sin\theta\cos\phi) + 0 = 0$$

$$g_{31} = \frac{\partial x}{\partial \phi}\frac{\partial x}{\partial r} + \frac{\partial y}{\partial \phi}\frac{\partial y}{\partial r} + \frac{\partial z}{\partial \phi}\frac{\partial z}{\partial r} = g_{13} = 0$$

$$g_{32} = \frac{\partial x}{\partial \phi}\frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial \phi}\frac{\partial y}{\partial \theta} + \frac{\partial z}{\partial \phi}\frac{\partial z}{\partial \theta} = g_{23} = 0$$

$$g_{33} = \frac{\partial x}{\partial \phi}\frac{\partial x}{\partial \phi} + \frac{\partial y}{\partial \phi}\frac{\partial y}{\partial \phi} + \frac{\partial z}{\partial \phi}\frac{\partial z}{\partial \phi} = (-r\sin\theta\sin\phi)^2 + (r\sin\theta\cos\phi)^2 + 0 = r^2\sin^2\theta, \qquad h_3 = r\sin\theta \tag{43}$$

The differential distance vector $d\mathbf{r}$ and the unit vector $\mathbf{e}_i$ may be given

$$d\mathbf{r} = h_1 dq_1\mathbf{e}_1 + h_2 dq_2\mathbf{e}_2 + h_3 dq_3\mathbf{e}_3 = \sum_{i=1}^{3} h_i q_i \mathbf{e}_i \tag{44}$$

$$\mathbf{e}_i = \frac{1}{h_i}\frac{\partial\mathbf{r}}{\partial q_i} \tag{45}$$

Here we only consider orthogonal curvlinear coordinates such as cylindrical coordinate and spherical coordinates.

The gradient can be defined by the sum of the product of the slope (rate of change) and the unit vector in the $i$-direction. The slope can be written as

$$\frac{\partial f}{\partial s_i} = \frac{1}{h_i}\frac{\partial f}{\partial q_i} \tag{46}$$

$$\tag{47}$$

Then the gradient in the orthogonal curvlinear coordinates

$$\mathrm{grad}_{\mathbf{q}} f = \nabla_{\mathbf{q}} f = \mathbf{e}_1\frac{\partial f}{\partial s_1} + \mathbf{e}_2\frac{\partial f}{\partial s_2} + \mathbf{e}_3\frac{\partial f}{\partial s_1} \tag{48}$$

$$= \mathbf{e}_1\frac{1}{h_1}\frac{\partial f}{\partial q_1} + \mathbf{e}_2\frac{1}{h_2}\frac{\partial f}{\partial q_2} + \mathbf{e}_3\frac{1}{h_3}\frac{\partial f}{\partial q_3} \tag{49}$$

$$\mathbf{e}_1 = \frac{1}{h_1}\frac{\partial\mathbf{r}}{\partial q_1}, \quad \mathbf{e}_2 = \frac{1}{h_2}\frac{\partial\mathbf{r}}{\partial q_2}, \quad \mathbf{e}_3 = \frac{1}{h_3}\frac{\partial\mathbf{r}}{\partial q_3} \tag{50}$$

In the cartesian coordinates,

$$\mathbf{q} = \{x, y, z\}, \quad \mathbf{r} = (x, y, z) \tag{51}$$

$$h_1 = 1, \quad \frac{\partial\mathbf{r}}{\partial x} = (1,0,0), \quad \mathbf{e}_1 = (1,0,0) = \mathbf{i} \tag{52}$$

$$h_2 = 1, \quad \frac{\partial\mathbf{r}}{\partial y} = (0,1,0), \quad \mathbf{e}_1 = (0,1,0) = \mathbf{j} \tag{53}$$

$$h_3 = 1, \quad \frac{\partial\mathbf{r}}{\partial z} = (0,0,1), \quad \mathbf{e}_1 = (0,0,1) = \mathbf{k} \tag{54}$$

$$\mathrm{grad} f = \nabla f = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) f \tag{55}$$

In the cylindrical coordinates,

$$\begin{aligned}
\mathbf{q} &= \{r,\theta,z\}, \quad \mathbf{r}=(x,y,z)=(r\cos\theta, r\sin\theta, z) && (56)\\
\frac{\partial \mathbf{r}}{\partial r} &= (\cos\theta,\sin\theta,0), \quad h_1 = \left|\frac{\partial \mathbf{r}}{\partial r}\right| = \sqrt{\cos^2\theta+\sin^2\theta}=1, \quad \mathbf{e}_1=(\cos\theta,\sin\theta,0) && (57)\\
\frac{\partial \mathbf{r}}{\partial \theta} &= (-r\sin\theta, r\cos\theta, 0), \quad h_2 = \left|\frac{\partial \mathbf{r}}{\partial \theta}\right| = r\sqrt{\cos^2\theta+\sin^2\theta}=r, \quad \mathbf{e}_2=(-\sin\theta,\cos\theta,0) && (58)\\
\frac{\partial \mathbf{r}}{\partial z} &= (0,0,1), \quad h_3 = \left|\frac{\partial \mathbf{r}}{\partial z}\right| = 1, \quad \mathbf{e}_3=(0,0,1) && (59)\\
\mathbf{e}_1\cdot\mathbf{e}_1 &= \cos^2\theta+\sin^2\theta=1, \mathbf{e}_2\cdot\mathbf{e}_2=\sin^2\theta+\cos^2\theta=1, \mathbf{e}_3\cdot\mathbf{e}_3=1 && (60)\\
\mathbf{e}_1\cdot\mathbf{e}_2 &= -\cos\theta\sin\theta+\sin\theta\cos\theta=0, \mathbf{e}_1\cdot\mathbf{e}_3=\mathbf{e}_2\cdot\mathbf{e}_3=0 && (61)\\
\mathbf{e}_i\cdot\mathbf{e}_j &= \delta_{i,j} && (62)\\
& \text{Here we used the Kronecker delta} && (63)\\
\delta_{i,j} &= \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases} && (64)\\
\mathbf{e}_1\times\mathbf{e}_2 &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \end{vmatrix} = \mathbf{k}(\cos^2\theta+\sin^2\theta)=\mathbf{e}_3 && (65)\\
\mathbf{e}_2\times\mathbf{e}_3 &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \mathbf{i}\cos\theta+\mathbf{j}\sin\theta=\mathbf{e}_1 && (66)\\
\mathbf{e}_3\times\mathbf{e}_1 &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ \cos\theta & \sin\theta & 0 \end{vmatrix} = -\mathbf{i}\sin\theta+\mathbf{j}\cos\theta=\mathbf{e}_2 && (67)\\
\mathbf{e}_i\times\mathbf{e}_j &= \epsilon_{i,j,k}\mathbf{e}_k && (68)\\
& \text{Here we use the Levi} - \text{Civita symbol} && (69)\\
\epsilon_{i,j,k} &= \begin{cases} 1 & (i,j,k)=(1,2,3),(2,3,1),(3,1,2) \\ -1 & (i,j,k)=(1,3,2),(2,1,3),(3,2,1) \\ 0 & \text{otherwise} \end{cases} && (70)\\
\mathrm{grad}_{r,\theta,z} f &= \nabla_{r,\theta,z} = \left(\mathbf{e}_1\frac{\partial}{\partial r}+\mathbf{e}_2\frac{1}{r}\frac{\partial}{\partial\theta}+\mathbf{e}_3\frac{\partial}{\partial z}\right) f && (71)
\end{aligned}$$

In the spherical coordinates,

$$\begin{aligned}
\mathbf{q} &= \{r,\theta,\phi\}, \quad \mathbf{r}=(x,y,z)=(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta) && (72)\\
\frac{\partial \mathbf{r}}{\partial r} &= (\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta), \quad h_1=\left|\frac{\partial \mathbf{r}}{\partial r}\right| = \sqrt{\sin^2\theta(\cos^2\phi+\sin^2\phi)+\cos^2\theta}=1, \\
& \mathbf{e}_1=(\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta) && (73)\\
\frac{\partial \mathbf{r}}{\partial \theta} &= (r\cos\theta\cos\phi, r\cos\theta\sin\phi, -r\sin\theta), \quad h_2=\left|\frac{\partial \mathbf{r}}{\partial \theta}\right| = r\sqrt{\cos^2\theta(\cos^2\phi+\sin^2\phi)+\sin^2\theta}=r \\
& \mathbf{e}_2=(\cos\theta\cos\phi,\cos\theta\sin\phi,-\sin\theta) && (74)\\
\frac{\partial \mathbf{r}}{\partial \phi} &= (-r\sin\theta\sin\phi, r\sin\theta\cos\phi, 0), \quad h_3=\left|\frac{\partial \mathbf{r}}{\partial \phi}\right| = r\sqrt{\sin^2\theta\sin^2\phi+\sin^2\theta\cos^2\phi}=r\sin\theta, \\
& \mathbf{e}_3=(-\sin\phi,\cos\phi,0) && (75)\\
\mathbf{e}_1\cdot\mathbf{e}_1 &= \sin^2\theta\cos^2\phi+\sin^2\theta\sin^2\phi+\cos^2\theta=1 && (76)\\
\mathbf{e}_2\cdot\mathbf{e}_2 &= \cos^2\theta\cos^2\phi+\cos^2\theta\sin^2\phi+\sin^2\theta=1 && (77)\\
\mathbf{e}_3\cdot\mathbf{e}_3 &= \sin^2\phi+\cos^2\phi=1 && (78)\\
\mathbf{e}_1\cdot\mathbf{e}_2 &= \sin\theta\cos\theta\cos^2\phi+\sin\theta\cos\theta\sin^2\phi-\sin\theta\cos\theta=0 && (79)\\
\mathbf{e}_2\cdot\mathbf{e}_3 &= -\cos\theta\sin\phi\cos\phi+\cos\theta\sin\phi\cos\phi+0=0 && (80)\\
\mathbf{e}_3\cdot\mathbf{e}_1 &= -\sin\theta\cos\phi\sin\phi+\sin\theta\cos\phi\sin\phi+0=0 && (81)
\end{aligned}$$

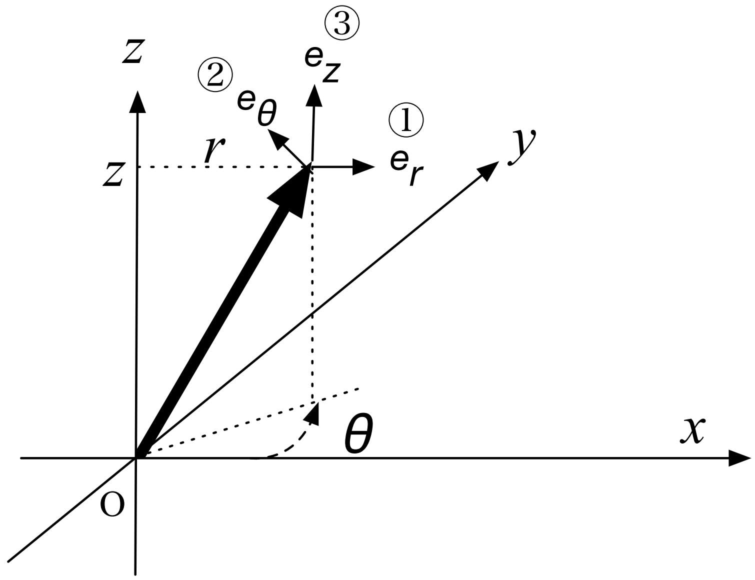


Figure 2: cylindrical coordinate

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{i,j} \tag{82}$$

$$\mathbf{e}_1 \times \mathbf{e}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \end{vmatrix} = \mathbf{i}(-\sin^2\theta\sin\phi - \cos^2\theta\sin\phi) \tag{83}$$

$$+\mathbf{j}(\cos^2\theta\cos\phi + \sin^2\theta\cos\phi) + \mathbf{k}(\sin\theta\cos\theta\sin\phi\cos\phi - \sin\theta\cos\theta\sin\phi\cos\phi) \tag{84}$$

$$= \mathbf{i}(-\sin\phi) + \mathbf{j}\cos\phi + \mathbf{k}0 = \mathbf{e}_3 \tag{85}$$

$$\mathbf{e}_2 \times \mathbf{e}_3 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{vmatrix} = \mathbf{i}(0 + \sin\theta\cos\phi) \tag{86}$$

$$+\mathbf{j}(\sin\theta\sin\phi + 0) + \mathbf{k}(\cos\theta\cos^2\phi + \cos\theta\sin^2\phi) \tag{87}$$

$$= \mathbf{i}\sin\theta\cos\phi + \mathbf{j}\sin\theta\sin\phi + \mathbf{k}\cos\theta = \mathbf{e}_1 \tag{88}$$

$$\mathbf{e}_3 \times \mathbf{e}_1 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin\phi & \cos\phi & 0 \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{vmatrix} = \mathbf{i}(\cos\theta\cos\phi + 0) \tag{89}$$

$$+\mathbf{j}(0 + \cos\theta\sin\phi) + \mathbf{k}(-\sin\theta\sin^2\phi - \sin\theta\cos^2\phi) \tag{90}$$

$$= \mathbf{i}\cos\theta\cos\phi + \mathbf{j}\cos\theta\sin\phi + \mathbf{k}(-\sin\theta) = \mathbf{e}_2 \tag{91}$$

$$\mathbf{e}_i \times \mathbf{e}_j = \epsilon_{i,j,k}\mathbf{e}_k \tag{92}$$

$$\mathrm{grad}_{r,\theta,\phi} f = \nabla_{r,\theta,\phi} = \left(\mathbf{e}_1\frac{\partial}{\partial r} + \mathbf{e}_2\frac{1}{r}\frac{\partial}{\partial\theta} + \mathbf{e}_3\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right) f \tag{93}$$

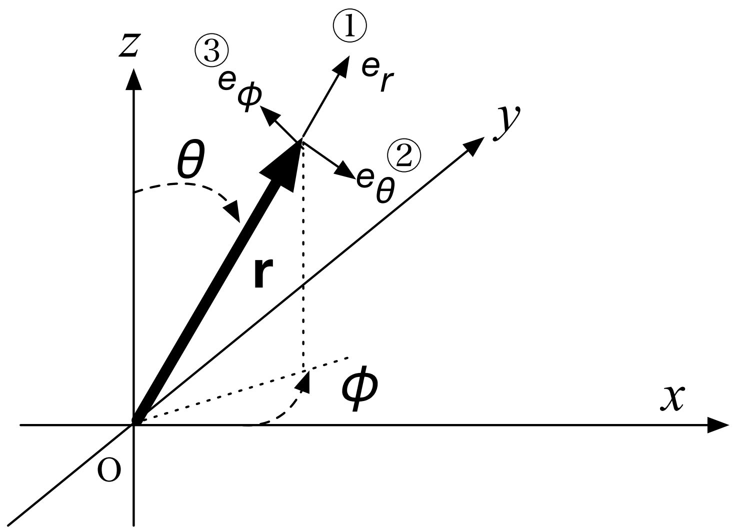


Figure 3: cylindrical coordinate