Transformation laws for tensors
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For a transformation from one orthogonal set of axises Ox_i to another Ox_i'

$$
\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{1}
$$

$$
x_i' = a_{ij} x_j \tag{2}
$$

Here a_{ij} is the cosine of the angle between x_i' and x_j. From this definition we cat get reverse transformation

$$
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} \tag{3}
$$

$$
x_i = a_{ji} x_j' \tag{4}
$$

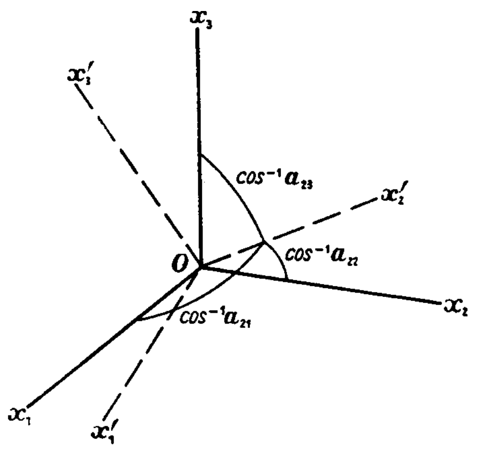

FIG. 1.3. Transformation of axes.

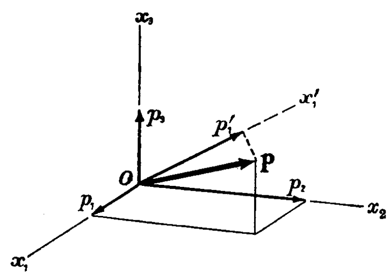

FIG. 1.4. Transformation of the components of a vector.

Figure 1:

Now we have a certain vector **p** whose components with respect to x_1, x_2, x_3 are p_1, p_2, p_3. This vector are written in new coordinate

$$
p_1' = p_1 \cos(\widehat{x_1 x_1'}) + p_2 \cos(\widehat{x_2 x_1'}) + p_3 \cos(\widehat{x_3 x_1'}) + \tag{5}
$$

Then we may write

$$\begin{pmatrix} p'_1 \\ p'_2 \\ p'_3 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} \quad (6)$$

$$p'_i = a_{ij} q_j \quad (7)$$

$$p_i = a_{ji} p'_j \quad (8)$$

If we consider the second-rank tensor(matrix), which relate the two vectors

$$\mathbf{p} = T\mathbf{q}, \qquad p_i = T_{ij} q_i \quad (9)$$

The transformations of the second-rank tensor becomes

$$p'_i = a_{ik} p_k \quad (10)$$

$$p_k = T_{kl} q_l \quad (11)$$

$$q_l = a_{jl} q'_j \quad (12)$$

$$p'_i = a_{ik} p_k = a_{ik} T_{kl} q_l = a_{ik} T_{kl} a_{jl} q'_j \quad (13)$$

$$\text{or} \quad p'_i = T'_{ij} q'_j \quad \text{then} \quad (14)$$

$$T'_{ij} = a_{ik} T_{kl} a_{jl} = a_{ik} a_{jl} T_{kl} \quad (15)$$

If we define the third-rank tensor which relate to the three vectors

$$p_i = \chi_{ijk} : q_j q_k \qquad \text{or} \mathbf{p} = \chi : \mathbf{qq} \quad (16)$$

The transformationd the third-rank tensor becomes

$$p'_i = a_{il} p_l \quad (17)$$

$$= a_{il} \chi_{lmn} q_m q_n \quad (18)$$

$$= a_{il} \chi_{lmn} a_{jm} q'_j a_{kn} q'_k \quad (19)$$

$$= a_{il} a_{jm} a_{kn} \chi_{lmn} q'_j q'_k \quad (20)$$

$$\text{or} = \chi'_{ijk} q'_j q'_k \quad \text{then} \quad (21)$$

$$\chi'_{ijk} = a_{il} a_{jm} a_{kn} \chi_{lmn} \quad (22)$$

If we consider the inversion symmetry (in a centrosymmetric media),

$$a_{ij} = -\delta_{ij} \quad (23)$$

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \quad (24)$$

The second-rank tensor χ_{ijk} becomes

$$\chi'_{ijk} = -\delta_{il} \delta_{jm} \delta_{kn} \chi_{lmn} \quad (25)$$

$$= -\chi_{ijk} \quad (26)$$

However the media has the centrosymmetric properties, then

$$\chi'_{ijk} = \chi_{ijk} \quad (27)$$

Therefore

$$\chi_{ijk} = 0 \quad (28)$$