Tayler expansion

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1 Let's consider a polynominal

If the function $f(x)$ can be approximated by the polynominals around $x = a$

$$f(x) \simeq a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + a_4(x-a)^4 + ... + a_n(x-a)^n \quad (1)$$

If $|x - a| << 1$, the above equations can be converged. We can get the 1st to n-th derivative such as

$$f'(x) = a_1 + 2a_2(x-a) + 3a_3(x-a)^2 + 4a_4(x-a)^3 + ... + na_n(x-a)^{n-1} \quad (2)$$

$$f''(x) = 2a_2 + 3 \times 2a_3(x-a) + 4 \times 3a_4(x-a)^2 + ... + n(n-1)a_n(x-a)^{n-2} \quad (3)$$

$$f'''(x) = 3 \times 2a_3 + 4 \times 3 \times 2a_4(x-a) + ... + n(n-1)(n-1)a_n(x-a)^{n-3} \quad (4)$$

$$f''''(x) = 4 \times 3 \times 2a_4 + ... + n(n-1)(n-2)(n-3)a_n(x-a)^{n-4} \quad (5)$$

$$f^{(n)}(x) = n!a_n \quad (6)$$

Then we have

$$a_1 = f'(a), \quad a_2 = \frac{1}{2!}f''(a), \quad a_3 = \frac{1}{3!}f'''(a), \quad a_4 = \frac{1}{4!}f''''(a), \quad a_n = \frac{1}{n!}f^{(n)}(a) \quad (7)$$

$$f(x) \simeq f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \frac{1}{3!}f'''(a)(x-a)^3 + ... + \frac{1}{n!}f^{(n)}(a)(x-a)^n \quad (8)$$

This is the Taylor expansion around $x = a$.

2 Let's consider more rigorous proof

The function $f(x)$ has a continuous n-th derivative at $a \leq x \leq b$. If we integrate the n-th derivative n times, we have

$$\begin{aligned}
\int_a^x dx f^{(n)}(x) &= f^{(n-1)}(x)\Big|_a^x = f^{(n-1)}(x) - f^{(n-1)}(a) \\
\int_a^x dx \left(\int_a^x dx f^{(n)}(x)\right) &= \int_a^x dx \left(f^{(n-1)}(x) - f^{(n-1)}(a)\right) \\
&= f^{(n-2)}(x) - f^{(n-2)}(a) - (x-a) f^{(n-1)}(a) \\
\int_a^x dx \left[\int_a^x dx \left(\int_a^x dx f^{(n)}(x)\right)\right] &= f^{(n-3)}(x) - f^{(n-3)}(a) - (x-a) f^{(n-2)}(a) - \frac{1}{2}(x-a)^2 f^{(n-1)}(a) \\
\int_a^x dx \left\{\int_a^x dx \left[\int_a^x dx \left(\int_a^x dx f^{(n)}(x)\right)\right]\right\} &= f^{(n-4)}(x) - f^{(n-4)}(a) - (x-a) f^{(n-3)}(a) - \frac{1}{2}(x-a)^2 f^{(n-2)}(a) \\
&\quad - \frac{1}{3 \times 2}(x-a)^3 f^{(n-1)}(a) \\
&\dots \\
\int_a^x \dots \int_a^x (dx)^n f^{(n)}(x) &= f(x) - f(a) - (x-a) f'(a) - \frac{1}{2!} f''(a)(x-a)^2 - \frac{1}{3!} f'''(a)(x-a)^3 \dots \\
&\quad - \frac{1}{(n-1)!}(x-a)^{n-1} f^{(n-1)}(a) \qquad (9) \\
& \qquad (10)
\end{aligned}$$

If we define $R_n = \int_a^x \dots \int_a^x (dx)^n f^{(n)}(x)$, we get the Taylor expansion

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2!} f''(a)(x-a)^2 + \frac{1}{3!} f'''(a)(x-a)^3 \dots + \frac{1}{(n-1)!} f^{(n-1)}(a)(x-a)^{n-1} + R_n \qquad (11)$$

In the case that $a = 0$ we get Maclaurin expansion

$$f(x) = f(0) + f'(0)x + \frac{1}{2!} f''(0)x^2 + \frac{1}{3!} f'''(0)x^3 \dots + \frac{1}{(n-1)!} f^{(n-1)}(0)x^{n-1} + R_n \qquad (12)$$

3 Examples

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 \dots + \frac{1}{n!}x^n + \dots \qquad (13)$$

$$e^{ix} = 1 + ix - \frac{1}{2!}x^2 - \frac{1}{3!}ix^3 + \frac{1}{4!}x^4 + \frac{1}{5!}ix^5 \dots \qquad (14)$$

$$\cos(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 \dots, \quad \sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 \dots, \quad \text{then we have} \quad e^{ix} = \cos x + i \sin x \qquad (15)$$