Electric field and charge at the corner or edge

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For simplicity Poisson equation is considered here, i.e. no charge except at the interface between the electrode and the medium.

$$\nabla^2\phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi = 0 \tag{1}$$

In the present work we will consider the potential and charge distribution at an edge or corner of the two conducting(metal) plate. We just follow the discussion done by J. D. Jackson's book on "Classical Electrodynamics, 3rd edition, Chap. 2, Section 11". In the Fig.1 the corner between the two conducting plate is shown.

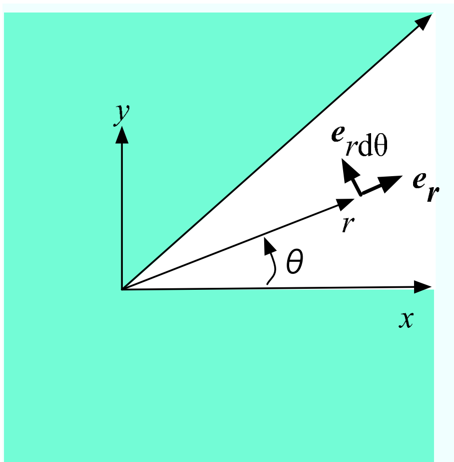

Figure 1:

It is convenient to use cylindrical coordinate r, θ, z. In the z direction the potential and charges are the same, then we will consider the r, θ problem.

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta}, \quad \frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} \tag{2}$$

$$r = \sqrt{x^2+y^2}, \quad \cos\theta = \frac{x}{r} = \frac{x}{\sqrt{x^2+y^2}} \tag{3}$$

$$\frac{\partial r}{\partial x} = \frac{1}{2}[x^2+y^2]^{-1/2}2x = \frac{x}{r} = \cos\theta, \quad \frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta \tag{4}$$

$$\frac{\partial \cos\theta}{\partial x} = \frac{\partial \cos\theta}{\partial \theta}\frac{\partial \theta}{\partial x} = -\sin\theta\frac{\partial \theta}{\partial x} = \frac{1}{r} + x\frac{-1}{2}\frac{1}{r^3}2x = \frac{1}{r}\left(1-\frac{x^2}{r^2}\right) = \frac{\sin^2\theta}{r}, \quad \frac{\partial \theta}{\partial x} = -\frac{\sin\theta}{r} \tag{5}$$

$$\frac{\partial \sin\theta}{\partial y} = \frac{\partial \sin\theta}{\partial \theta}\frac{\partial \theta}{\partial y} = \cos\theta\frac{\partial \theta}{\partial y} = \frac{1}{r} + y\frac{-1}{2}\frac{1}{r^3}2y = \frac{1}{r}\left(1-\frac{y^2}{r^2}\right) = \frac{\cos^2\theta}{r}, \quad \frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r} \tag{6}$$

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}, \quad \frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta} \tag{7}$$

$$
\begin{aligned}
\frac{\partial^2}{\partial x^2} &= \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right) = \cos^2\theta\frac{\partial^2}{\partial r^2} + \frac{\cos\theta\sin\theta}{r^2}\frac{\partial}{\partial\theta} - \frac{\cos\theta\sin\theta}{r}\frac{\partial^2}{\partial r\partial\theta} \\
&\quad + \frac{\sin^2\theta}{r}\frac{\partial}{\partial r} - \frac{\sin\theta\cos\theta}{r}\frac{\partial^2}{\partial r\partial\theta} + \frac{\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2} \qquad (8) \\
\frac{\partial^2}{\partial y^2} &= \left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right) = \sin^2\theta\frac{\partial^2}{\partial r^2} - \frac{\cos\theta\sin\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{\cos\theta\sin\theta}{r}\frac{\partial^2}{\partial r\partial\theta} \\
&\quad + \frac{\cos^2\theta}{r}\frac{\partial}{\partial r} + \frac{\sin\theta\cos\theta}{r}\frac{\partial^2}{\partial r\partial\theta} - \frac{\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{\cos^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2} \qquad (9) \\
\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} &= \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} \qquad (10) \\
\nabla^2\phi(r,\theta) &= \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\phi(r,\theta) = 0 \qquad (11)
\end{aligned}
$$

The radial direction $\mathbf{e_r}$ and the angular directon $\mathbf{e}_{rd\theta}$ as shown in Fig.1 is perpendicular, then the gradient of the potential ϕ can be defined

$$
\nabla\phi = \mathbf{e_r}\frac{\partial}{\partial r} + \mathbf{e}_{rd\theta}\frac{1}{r}\frac{\partial}{\partial\theta} \qquad (12)
$$

If we can separate the solution in r and θ variables,

$$
\begin{aligned}
\phi(r,\theta) &= R(r)\Theta(\theta) \qquad (13) \\
\nabla^2\phi(r,\theta) &= \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\phi(r,\theta) = \Theta\left(\frac{\partial^2 R}{\partial r^2} + \frac{1}{r}\frac{\partial R}{\partial r}\right) + \frac{R}{r^2}\frac{\partial^2\Theta}{\partial\theta^2} = 0 \qquad (14) \\
\frac{r^2}{R}\left(\frac{\partial^2 R}{\partial r^2} + \frac{1}{r}\frac{\partial R}{\partial r}\right) &= -\frac{1}{\Theta}\frac{\partial^2\Theta}{\partial\theta^2} = \omega^2 \qquad (15)
\end{aligned}
$$

The general solutions for $\omega \neq 0$ are

$$
\begin{aligned}
R(r) &= ar^{\omega} + br^{-\omega} \qquad (16) \\
\Theta(\theta) &= c\cos(\omega\theta) + d\sin(\omega\theta) \qquad (17)
\end{aligned}
$$

and for $\omega = 0$

$$
\begin{aligned}
R(r) &= \alpha + \beta\ln r \qquad (18) \\
\Theta(\theta) &= \gamma + \delta\theta \qquad (19)
\end{aligned}
$$

From the boundary conditions, i.e. $\phi(r,\theta=0) = V, \phi(r,\theta=\theta_0) = V$

$$
c = 0,\ [\sin 0 = 0, \sin(\omega\theta_0) = 0], \quad \delta = 0 \qquad (20)
$$

$r = 0$ is included, then

$$
b = 0, \quad \beta = 0 \qquad (21)
$$

and

$$
\sin(\omega\theta_0) = 0, \quad \omega = \frac{n\pi}{\theta_0}, \quad n = 1, 2, 3, ... \qquad (22)
$$

Then we have

$$
\phi(r,\theta) = V + \sum_{n=1}^{\infty} a_n r^{n\pi/\theta_0}\sin(n\pi\theta/\theta_0) \qquad (23)
$$

Near $r = 0$ the $n = 1$ term is important, then we can write

$$
\phi(r,\theta) \simeq V + a_1 r^{\pi/\theta_0}\sin(\pi\theta/\theta_0) \qquad (24)
$$

The electric field near $r = 0$ is given by

$$
\begin{aligned}
\mathbf{E} &= -\nabla\phi = -\mathbf{e_r}\frac{\partial}{\partial r} - \mathbf{e}_{rd\theta}\frac{1}{r}\frac{\partial}{\partial\theta} && (25)\\
\mathbf{E}_r &\simeq -a_1\frac{\pi}{\theta_0}r^{\pi/\theta_0-1}\sin(\pi\theta/\theta_0) && (26)\\
\mathbf{E}_{rd\theta} &= -a_1\frac{\pi}{\theta_0}r^{\pi/\theta_0-1}\cos(\pi\theta/\theta_0) && (27)
\end{aligned}
$$

From the Gauss theorem at the surface, we can write with the unit vector $\mathbf{n}$ form metal to vacuum

$$\mathbf{n}\cdot[\mathbf{D}_{\text{vac}} - \mathbf{D}_{\text{metal}}] = \sigma_{\text{metal}|\text{vac}}, \quad \mathbf{D}_{\text{metal}} = 0$$

The surface charge density at $\phi = 0$ and $\phi = \theta_0$ are equal and can be approximated

$$\sigma(r) = \mathbf{D}_{\text{vac}} = \epsilon\epsilon_0\mathbf{E}_{rd\theta} = -a_1\frac{\pi\epsilon\epsilon_0}{\theta_0}r^{\pi/\theta_0-1} \tag{28}$$

When θ_0 is small, the power of r becomes large and no charge accumulation at the corner $r = 0$. For flat surface $\theta_0 = \pi$, there is no dependence on r. When $\theta_0 > \pi$ the power $\pi/\theta_0 - 1$ of r becomes negative, and at the edge the surface charge density become singular.

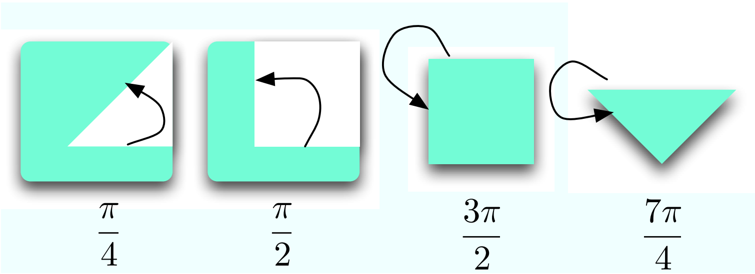

Figure 2: coner and edge with $\theta_0 = \pi/4, \pi/2, 3\pi/2, 7\pi/4$.

Table 1: r sigularity at the coner or edge

θ_0	$\pi/\theta_0 - 1$	r^{π/θ_0-1}
0	$+\infty$	
$\pi/4$	3	r^3
$\pi/2$	1	r
π	0	1
$3\pi/2$	-1/3	$r^{-1/3}$
$7\pi/4$	-3/7	$r^{-3/7}$
2π	-1/2	$r^{-1/2}$