18 章章末問題

問題 1

$[\hat{P}_x, \hat{X}] = -i\hbar\hat{I}$ であるが，$[\hat{X}, \hat{P}_x]$はどうなるか？

解答

$$\hat{X}\hat{P}_x\psi = x(-i\hbar\frac{\mathrm{d}}{\mathrm{d}x})\psi, \hat{P}_x\hat{X}\psi = -i\hbar\frac{\mathrm{d}}{\mathrm{d}x}(x\psi) = -i\hbar(1+x\frac{\mathrm{d}}{\mathrm{d}x})\psi$$

$$[\hat{X}\hat{P}_x - \hat{P}_x\hat{X}]\psi = +i\hbar\psi, [\hat{X}, \hat{P}_x] = [\hat{X}\hat{P}_x - \hat{P}_x\hat{X}] = +i\hbar\hat{I}$$

問題 2

$[\hat{X}, \hat{X}], [\hat{X}, \hat{Y}], [\hat{X}, \hat{Z}], [\hat{P}_y, \hat{Y}], [\hat{Y}, \hat{P}_y], [\hat{Y}, \hat{P}_x][\hat{P}_x, \hat{Y}], [\hat{P}_x, \hat{Z}], [\hat{P}_x, \hat{P}_x], [\hat{P}_x, \hat{P}_y], [\hat{P}_x, \hat{P}_z]$
はどうなるのか上の交換関係を波動関数に演算して求めよ。

解答

$[\hat{P}_y, \hat{Y}] = -i\hbar\hat{I}, [\hat{Y}, \hat{P}_y] = +i\hbar\hat{I}, [\hat{P}_z, \hat{Z}] = -i\hbar\hat{I}, [\hat{Z}, \hat{P}_z] = +i\hbar\hat{I}$
これ以外はすべてゼロ（可換）である。

問題 3　WEB をみて Shcwarz の不等式を証明せよ。

解答（略 WEB を見よ）

問題 4

$[\hat{L}_y, \hat{L}_x] = -i\hbar\hat{L}_z, [\hat{L}_y, \hat{L}_z] = i\hbar\hat{L}_x, [\hat{L}_z, \hat{L}_y] = -i\hbar\hat{L}_x, [\hat{L}_z, \hat{L}_x] = i\hbar\hat{L}_y, [\hat{L}_x, \hat{L}_z] = -i\hbar\hat{L}_y$
を証明せよ。

解答

$$[\hat{L}_y, \hat{L}_x] = \hat{L}_y\hat{L}_x - \hat{L}_x\hat{L}_y = (\hat{Z}\hat{P}_x - \hat{X}\hat{P}_z)(\hat{Y}\hat{P}_z - \hat{Z}\hat{P}_y) - (\hat{Y}\hat{P}_z - \hat{Z}\hat{P}_y)(\hat{Z}\hat{P}_x - \hat{X}\hat{P}_z)$$

$$= -\hat{Y}\hat{P}_z\hat{Z}\hat{P}_x + \hat{Y}\hat{P}_z\hat{X}\hat{P}_z + \hat{Z}\hat{P}_y\hat{Z}\hat{P}_x - \hat{Z}\hat{P}_y\hat{X}\hat{P}_z + \hat{Z}\hat{P}_x\hat{Y}\hat{P}_z - \hat{Z}\hat{P}_x\hat{Z}\hat{P}_y - \hat{X}\hat{P}_z\hat{Y}\hat{P}_z + \hat{X}\hat{P}_z\hat{Z}\hat{P}_y$$

$$= -\hat{Y}\hat{P}_z\hat{Z}\hat{P}_x + \hat{Y}\hat{Z}\hat{P}_z\hat{P}_x - \hat{X}\hat{Z}\hat{P}_z\hat{P}_y + \hat{X}\hat{P}_z\hat{Z}\hat{P}_y = -\hat{Y}[\hat{P}_z, \hat{Z}]\hat{P}_x - \hat{X}[\hat{Z}, \hat{P}_z]\hat{P}_y$$

$$= -i\hbar(-\hat{Y}\hat{P}_x + \hat{X}\hat{P}_y) = -i\hbar\hat{L}_z$$

$$[\hat{L}_y, \hat{L}_z] = \hat{L}_y\hat{L}_z - \hat{L}_z\hat{L}_y = (\hat{Z}\hat{P}_x - \hat{X}\hat{P}_z)(\hat{X}\hat{P}_y - \hat{Y}\hat{P}_x) - (\hat{X}\hat{P}_y - \hat{Y}\hat{P}_x)(\hat{Z}\hat{P}_x - \hat{X}\hat{P}_z)$$

$$= \hat{Z}\hat{P}_x\hat{X}\hat{P}_y - \hat{Z}\hat{P}_x\hat{Y}\hat{P}_x - \hat{X}\hat{P}_z\hat{X}\hat{P}_y + \hat{X}\hat{P}_z\hat{Y}\hat{P}_x - \hat{X}\hat{P}_y\hat{Z}\hat{P}_x + \hat{X}\hat{P}_y\hat{X}\hat{P}_z + \hat{Y}\hat{P}_x\hat{Z}\hat{P}_x - \hat{Y}\hat{P}_x\hat{X}\hat{P}_z$$

$$= \hat{Z}[\hat{P}_x\hat{X} - \hat{X}\hat{P}_x]\hat{P}_y + \hat{Y}[\hat{X}\hat{P}_x - \hat{P}_x\hat{X}]\hat{P}_z = i\hbar(-\hat{Z}\hat{P}_y + \hat{Y}\hat{P}_z) = i\hbar\hat{L}_x$$

$$[\hat{L}_z, \hat{L}_y] = \hat{L}_z\hat{L}_y - \hat{L}_y\hat{L}_z = -[\hat{L}_y, \hat{L}_z] = -i\hbar\hat{L}_x$$

$$[\hat{L}_z,\hat{L}_x]=(\hat{X}\hat{P}_y-\hat{Y}\hat{P}_x)(\hat{Y}\hat{P}_z-\hat{Z}\hat{P}_y)-(\hat{Y}\hat{P}_z-\hat{Z}\hat{P}_y)(\hat{X}\hat{P}_y-\hat{Y}\hat{P}_x)$$
$$=\hat{X}\hat{P}_y\hat{Y}\hat{P}_z-\hat{X}\hat{P}_y\hat{Z}\hat{P}_y-\hat{Y}\hat{P}_x\hat{Y}\hat{P}_z+\hat{Y}\hat{P}_x\hat{Z}\hat{P}_y-\hat{Y}\hat{P}_z\hat{X}\hat{P}_y+\hat{Y}\hat{P}_z\hat{Y}\hat{P}_x+\hat{Z}\hat{P}_y\hat{X}\hat{P}_y-\hat{Z}\hat{P}_y\hat{Y}\hat{P}_x$$
$$=\hat{Z}[\hat{Y}\hat{P}_y-\hat{P}_y\hat{Y}]\hat{P}_x+\hat{X}[\hat{P}_y\hat{Y}-\hat{Y}\hat{P}_y]\hat{P}_z=i\hbar(\hat{Z}\hat{P}_x-\hat{X}\hat{P}_z)=i\hbar\hat{L}_y$$
$$[\hat{L}_x,\hat{L}_z]=-[\hat{L}_z,\hat{L}_x]=-i\hbar\hat{L}_y$$

問題 5　　以下の交換関係を波動関数に演算して求めよ。

$$[\hat{X},\hat{L}_x]=0,[\hat{X},\hat{L}_y]=i\hbar\hat{Z},[\hat{X},\hat{L}_z]=-i\hbar\hat{Y}$$
$$[\hat{P}_x,\hat{L}_x]=0,[\hat{P}_x,\hat{L}_y]=i\hbar\hat{P}_z,[\hat{P}_x,\hat{L}_z]=-i\hbar\hat{P}_y$$
$$[\hat{X},\hat{L}^2]=i\hbar(\hat{L}_y\hat{Z}+\hat{Z}\hat{L}_y-\hat{L}_z\hat{Y}-\hat{Y}\hat{L}_z)$$
$$[\hat{P}_x,\hat{L}^2]=i\hbar(\hat{L}_y\hat{P}_z+\hat{P}_z\hat{L}_y-\hat{L}_z\hat{P}_y-\hat{P}_y\hat{L}_z)$$

解答

$$[\hat{X},\hat{L}_x]=[\hat{X},\hat{Y}\hat{P}_z-\hat{Z}\hat{P}_y]=\hat{X}\hat{Y}\hat{P}_z-\hat{X}\hat{Z}\hat{P}_y-\hat{Y}\hat{P}_z\hat{X}+\hat{Z}\hat{P}_y\hat{X}=0$$
$$[\hat{X},\hat{L}_y]=[\hat{X},\hat{Z}\hat{P}_x-\hat{X}\hat{P}_z]=\hat{X}\hat{Z}\hat{P}_x-\hat{X}\hat{X}\hat{P}_z-\hat{Z}\hat{P}_x\hat{X}+\hat{X}\hat{P}_z\hat{X}=\hat{Z}[\hat{X},\hat{P}_x]=i\hbar\hat{Z}$$
$$[\hat{X},\hat{L}_z]=[\hat{X},\hat{X}\hat{P}_y-\hat{Y}\hat{P}_x]=\hat{X}\hat{X}\hat{P}_y-\hat{X}\hat{Y}\hat{P}_x-\hat{X}\hat{P}_y\hat{X}+\hat{Y}\hat{P}_x\hat{X}=\hat{Y}[\hat{P}_x,\hat{X}]=-i\hbar\hat{Y}$$
$$[\hat{P}_x,\hat{L}_x]=[\hat{P}_x,\hat{Y}\hat{P}_z-\hat{Z}\hat{P}_y]=\hat{P}_x\hat{Y}\hat{P}_z-\hat{P}_x\hat{Z}\hat{P}_y-\hat{Y}\hat{P}_z\hat{P}_x+\hat{Z}\hat{P}_y\hat{P}_x=0$$
$$[\hat{P}_x,\hat{L}_y]=[\hat{P}_x,\hat{Z}\hat{P}_x-\hat{X}\hat{P}_z]=\hat{P}_x\hat{Z}\hat{P}_x-\hat{P}_x\hat{X}\hat{P}_z-\hat{Z}\hat{P}_x\hat{P}_x+\hat{X}\hat{P}_z\hat{P}_x=[\hat{X},\hat{P}_x]\hat{P}_z=i\hbar\hat{P}_z$$
$$[\hat{P}_x,\hat{L}_z]=[\hat{P}_x,\hat{X}\hat{P}_y-\hat{Y}\hat{P}_x]=\hat{P}_x\hat{X}\hat{P}_y-\hat{P}_x\hat{Y}\hat{P}_x-\hat{X}\hat{P}_y\hat{P}_x+\hat{Y}\hat{P}_x\hat{P}_x=[\hat{P}_x,\hat{X}]\hat{P}_y=-i\hbar\hat{P}_y$$
$$[\hat{X},\hat{L}^2]=[\hat{X},\hat{L}_x^2+\hat{L}_y^2+\hat{L}_z^2]=\hat{X}\hat{L}_x^2+\hat{X}\hat{L}_y^2+\hat{X}\hat{L}_z^2-\hat{L}_x^2\hat{X}-\hat{L}_y^2\hat{X}-\hat{L}_z^2\hat{X}$$
$$=\hat{X}\hat{L}_x\hat{L}_x-\hat{L}_x\hat{L}_x\hat{X}+\hat{X}\hat{L}_y\hat{L}_y-\hat{L}_y\hat{L}_y\hat{X}+\hat{X}\hat{L}_z\hat{L}_z-\hat{L}_z\hat{L}_z\hat{X}$$
$$=\hat{L}_x\hat{X}\hat{L}_x-\hat{L}_x\hat{X}\hat{L}_x+(i\hbar\hat{Z}+\hat{L}_y\hat{X})\hat{L}_y-\hat{L}_y(\hat{X}\hat{L}_y-i\hbar\hat{Z})+(-i\hbar\hat{Y}+\hat{L}_z\hat{X})\hat{L}_z-\hat{L}_z(i\hbar\hat{Y}+\hat{X}\hat{L}_z)$$
$$=i\hbar(\hat{L}_y\hat{Z}+\hat{Z}\hat{L}_y-\hat{L}_z\hat{Y}-\hat{Y}\hat{L}_z)$$
$$[\hat{P}_x,\hat{L}^2]=[\hat{P}_x,\hat{L}_x^2+\hat{L}_y^2+\hat{L}_z^2]=\hat{P}_x\hat{L}_x^2+\hat{P}_x\hat{L}_y^2+\hat{P}_x\hat{L}_z^2-\hat{L}_x^2\hat{P}_x-\hat{L}_y^2\hat{P}_x-\hat{L}_z^2\hat{P}_x$$
$$=\hat{P}_x\hat{L}_x\hat{L}_x-\hat{L}_x\hat{L}_x\hat{P}_x+\hat{P}_x\hat{L}_y\hat{L}_y-\hat{L}_y\hat{L}_y\hat{P}_x+\hat{P}_x\hat{L}_z\hat{L}_z-\hat{L}_z\hat{L}_z\hat{P}_x$$
$$=\hat{L}_x\hat{P}_x\hat{L}_x-\hat{L}_x\hat{P}_x\hat{L}_x+(i\hbar\hat{P}_z+\hat{L}_y\hat{P}_x)\hat{L}_y-\hat{L}_y(\hat{P}_x\hat{L}_y-i\hbar\hat{P}_z)+(-i\hbar\hat{P}_y+\hat{L}_z\hat{P}_x)\hat{L}_z-\hat{L}_z(i\hbar\hat{P}_y+\hat{P}_x\hat{L}_z)$$
$$=i\hbar(\hat{L}_y\hat{P}_z+\hat{P}_z\hat{L}_y-\hat{L}_z\hat{P}_y-\hat{P}_y\hat{L}_z)$$